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Non-axisymmetric instability of a rotating sheet of gas in a rotating environment

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We discuss non-axisymmetric instability of a thin cylindrical sheet of a gas which rotates concentrically with a rigidly rotating environment. For the sake of simplicity, we restrict ourselves to a linear single-mode analysis of a two-dimensional disturbance for which the axial component of the wavenumber vector vanishes. We further restrict ourselves to two limiting cases. In case 1 the gas can be treated as incompressible, while in case 2 the effect of radial stratification caused by the centrifugal force is extremely strong. In case 1 there are two unstable modes: a travelling and a stationary disturbance with respect to a system of co-ordinates which rotates with the environment. For each disturbance, we show the domains of instability on ρ *vs.* ω diagrams, where ρ is the density and ω is the angular velocity of the sheet non-dimensionalized with respect to those of the environment. A negative-viscosity phenomenon is also described. In case 2 both the travelling and the stationary disturbances are stabilized by the strong radial stratification. An outline of a WKB method of approximation is given.

1. Introduction and summary

In their study of instability of swirling flow, Howard & Gupta (1962) described many interesting aspects, including a semicircle theorem. They did not discuss cases in which there are discontinuities in the flow, because instability is then automatically expected. Followers of their study also disregarded discontinuous configurations (see, for example, Pedley 1968, 1969; Maslowe 1974; Gans 1975).

Discontinuities in the angular velocity in a rotating fluid are smoothed out by Stewartson layers. The instability of Stewartson layers poses interesting problems, which are not included in the theory of continuous flows. Thus Busse (1968), Siegmann (1974) and Hashimoto (1976) discussed the instability of Stewartson layers in relation to Hide & Titman's (1967) experiment. If there are two discontinuities in the angular velocity, as for a cylindrical sheet in a rotating fluid, other interesting phenomena appear. This kind of cylindrical sheet in a rotating fluid occurs in gas centrifuges used for the enrichment of uranium.

Let us consider a gas centrifuge in which the operating gas is fed in through a narrow circular slit in the lid. As for steady flows (see, for example, Matsuda, Sakurai & Takeda 1975; Nakayama & Usui 1974; Hashimoto 1975), half of this gas forms a thin cylindrical sheet with the slit as its top cross-section. In these

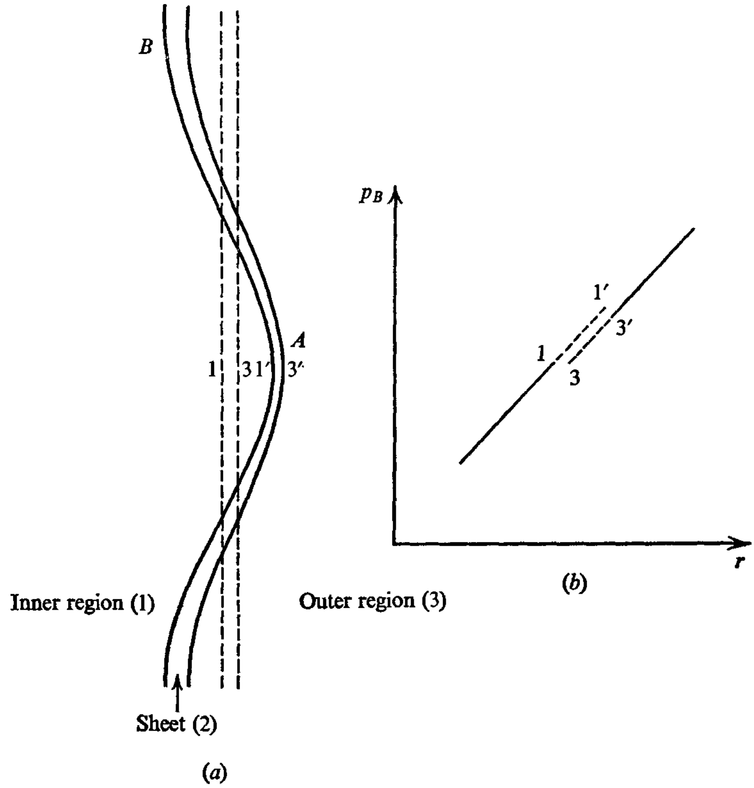

FIGURE 1. Possible interaction between the sheet and the environment. (*a*) Schematic diagram of non-axisymmetric two-dimensional disturbance on the sheet. Regions 1, 3 and 2 are the inner and the outer parts of the environment and the sheet, respectively. In this figure the sheet is assumed not to rotate. A part A of the sheet has moved outwards and is being squeezed by a high pressure at its new location. (*b*) The variation of the pressure of the inner and outer environment on the interfaces estimated by linear extrapolation or interpolation of the basic pressure. (In both parts of the figure points 1, 1′ and 3, 3′ designate the original and the new locations of the inner and the outer interfaces of part A respectively.) The resultant pressure of the environment at the new location of part A is clearly in the outward direction. This causes further outward motion of part A.

treatments, differences between the density and angular velocity of the gas fed in and the gas which occupies the rest of the centrifuge, respectively, are assumed to be infinitesimally small. In practice, this assumption is violated to a considerable extent. In such circumstances many kinds of instability are excited, as was expected by Howard & Gupta (1962). For example, Rayleigh–Taylor instability is excited owing to the density discontinuity if the wavelength of the disturbance is small in comparison with the thickness of the sheet, since the sheet can then be considered to be of infinite thickness. However this is not the case if the wavelength of the disturbance is large in comparison with the thickness of the sheet. Figure 1 gives a schematic representation of such a disturbance. If the density in the inner region 1 were larger than that in the outer region 3, the potential energy of the system would be decreased by this disturbance. This loss of potential energy would then be transformed into kinetic energy of the disturbance. This is the mechanism by which the Rayleigh–Taylor instability is excited. In the present case, this mechanism does not operate because the inner and the outer densities are equal. The augmentation of the disturbance depends in this case upon a subtle interaction between the sheet and the environment. To illustrate a possible interaction, let us consider part A of the sheet, which is displaced outwards by the disturbance. For the sake of definiteness, we assume that the sheet does not rotate. Because of the high pressure at its new location, part A is squeezed and its thickness is decreased. If the environmental pressure is estimated by linear extrapolation or interpolation of the basic pressure, this decrease in the thickness leads to an outward resultant pressure on part A (see figure 1*b*). This outward pressure causes further outward motion of part A. Augmentation of the inward displacement of part B can be conjectured similarly.

By the way, the squeezing of part A forces gas out of that part of the sheet. This motion, however, is obstructed by the centrifugal force if the sheet rotates. A disturbance pressure related to the gas motion which causes the displacement itself may compensate for the change in the basic pressure. We have neglected these counteracting effects in the above illustrative example. The best way to include all these effects is to treat the problem analytically. This is the purpose of this paper.

As is suggested by the above illustration, the predominant effect is interaction between the sheet and the environment as a two-dimensional configuration. Axial motion of the sheet seems to have a minor influence. This corresponds to practical cases, because the height of a centrifuge is large in comparison with its radius. It is also reasonable to expect that the axial component of the sheet velocity will be small in comparison with the azimuthal component for a gas centrifuge, so we assume that the axial component vanishes. For the sake of simplicity, we neglect the top and the bottom end plates and assume that the axial component of the wavenumber vector vanishes. We also neglect the viscosity and the thermal conductivity of the gas. The effect of the side walls is neglected whenever this simplifies the treatment.

Summarizing the above statements, we formulate our problem as follows. A thin cylindrical sheet of inviscid non-conducting gas rotates rigidly in and concentrically with a rigidly rotating environment. The environment is also inviscid and non-conducting and its angular velocity is different from that of the sheet. The axial components of the velocities of the sheet and the environment both vanish. In this basic state of equilibrium, the centrifugal force is balanced by the pressure. A small amplitude disturbance, the axial component of whose wavenumber vector vanishes, is superposed on this equilibrium state. Our problem is to study analytically whether or not this disturbance grows. For the sake of simplicity, we restrict ourselves to two limiting cases. In case 1, the operating gas is treated as incompressible, the densities of the sheet and the environment are prescribed and the side walls are neglected. In case 2, the effect of the radial density stratification is extremely strong. This situation corresponds to the operating conditions for gas centrifuges used for the enrichment of uranium. Also, the sheet and the environment are individually isothermal, and the side walls are taken into account for the sake of definiteness.

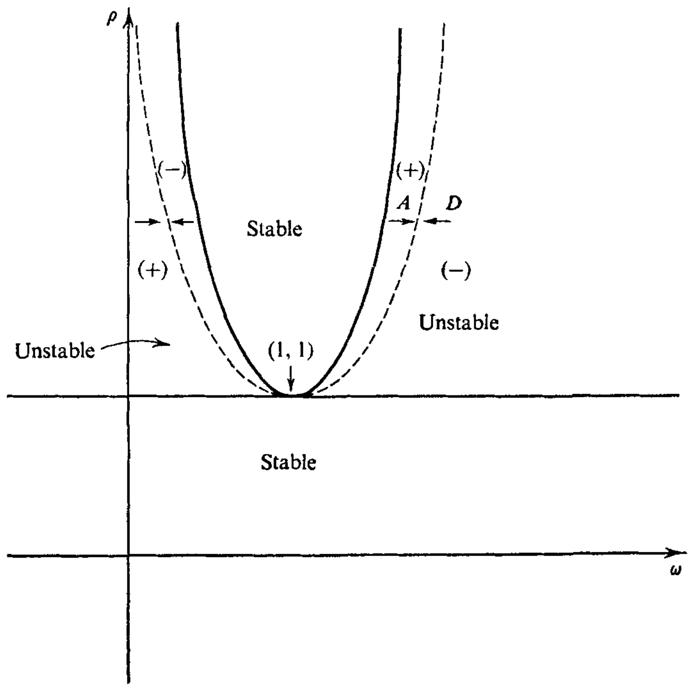

FIGURE 2. Domains of instability of the stationary disturbance with $m = 1$. The restriction to this value of m is based on the fact that this disturbance is the most stable and its domains of instability give us the union of those corresponding to other (stationary) disturbances. The signs of angular-momentum transfer to the sheet via the disturbance are also shown, a plus sign meaning that the sheet is accelerated. Trends of induced changes in the angular velocity of the sheet are indicated by arrows. Dashed lines are boundaries on which the angular-momentum transfer changes sign. As explained in the text, this is the line to which the angular velocity of the sheet tends asymptotically.

Before describing the mathematical details, we want to summarize the results. In case 1, the analysis is simple, and we give it in full detail in §§2 and 3. We show that there are two kinds of unstable disturbance: one travelling and the other stationary with respect to a system of co-ordinates which rotates with the environment. As we shall show in §3, all the disturbances are unstable if the disturbance with $m = 1$ is unstable, where the integer m is the azimuthal component of the wavenumber vector. Figures 2 and 3 show the domains of instability on ρ *vs.* ω diagrams of this disturbance, where ρ and ω are respectively the density and angular velocity of the sheet non-dimensionalized with respect to those of the environment. In these figures we give the directions of angular-momentum transfer via the disturbance, a plus sign corresponding to acceleration of the rotation of the sheet.

It should be noted that there are domains of instability in the immediate neighbourhood of the point $(1, 1)$. This point corresponds to a uniform configuration. Therefore sheet instability can appear when the basic state of rotation is only very slightly non-uniform.

To see another interesting aspect, let us consider the evolution of a state of

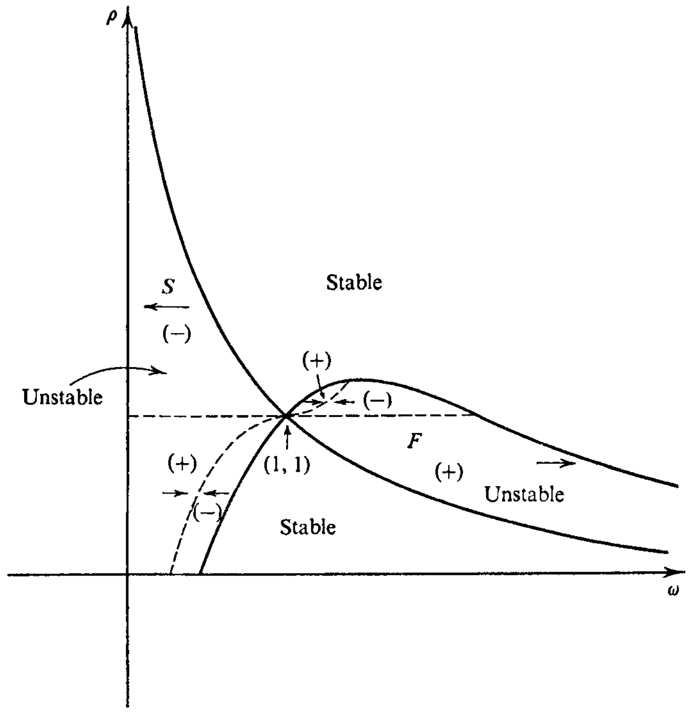

FIGURE 3. Domains of instability of the travelling disturbance with $m = 1$. The format of the representation is the same as that in figure 2. The interesting thing here is that the sheet tends to stop rotating in region S because of the angular-momentum transfer via the disturbance. It is also interesting that in region F the sheet is accelerated until its angular velocity reaches that on the right boundary of the region.

given ρ according to figure 2. Region A of this figure is labelled by a plus sign, which corresponds to acceleration of the sheet by angular-momentum transfer via the disturbance. Thus a state of given ρ moves to the right in this region until a dashed line on the diagram is reached. This line is a boundary on which the angular-momentum transfer changes sign. Similarly, a state in region D of figure 2 moves to the left and approaches the same dashed line. Points on this dashed line therefore correspond to an asymptotic state to which the angular velocity of a sheet with a prescribed density tends. Other asymptotic states are indicated by the other dashed lines in figures 2 and 3. In particular, the asymptotic state to which states in region S in figure 3 tend is a state of no rotation of the sheet. The asymptotic state corresponding to region F in figure 3 is the right boundary of the domain. In region A in figure 2, the disturbance augments the angular-velocity difference between the sheet and the environment. This tendency is in contrast to that of the viscosity, which acts to diminish the difference in velocity. This tendency is a manifestation of the negative-viscosity phenomena familiar in meteorology (Starr 1968).

In case 2, a WKB method of approximation similar to that applied to the baroclinic-type instability in a gas centrifuge (Sakurai 1975) is applied. Because the analysis is tedious though straightforward, we shall merely give an outline of it in §4. The result is that both travelling and stationary disturbances are stabilized by strong radial stratification caused by the centrifugal force. As far as sheet instability is concerned, this result may be favourable for gas centrifuges used for the enrichment of uranium.

Finally, it should be noted that the exact locations of the asymptotic states mentioned above must be determined by a nonlinear treatment. Coupling between modes may play an important role. The contrasting results for cases 1 and 2 also make it natural to inquire about the situation in an intermediate case. These problems are, however, outside the scope of the present treatment.

2. Basic equations

In this section, we restrict ourselves to case 1, because extension of the procedure used here to case 2 is straightforward. The latter will be discussed briefly in §4.

We use a system of cylindrical co-ordinates which rotates with the environment. We define the following non-dimensional variables:

$$\left.\begin{aligned} &r = \tilde{r}/\tilde{r}_0, \quad z = \tilde{z}/\tilde{r}_0, \quad \phi = \tilde{\phi} - \tilde{\omega}_0 \tilde{t}, \quad t = \tilde{\omega}_0 \tilde{t},\\ &u = \frac{\tilde{q}_r}{\epsilon \tilde{r}_0 \tilde{\omega}_0}, \quad v = \frac{\tilde{q}_\phi - \omega \tilde{r} \tilde{\omega}_0}{\epsilon \tilde{r}_0 \tilde{\omega}_0}, \quad w = \frac{\tilde{q}_z}{\epsilon \tilde{r}_0 \tilde{\omega}_0},\\ &p = (\tilde{p} - \tilde{\rho}_0 \tilde{\omega}_0^2 \tilde{r}_0^2 p_B)/(\epsilon \tilde{\rho}_0 \tilde{\omega}_0^2 \tilde{r}_0^2), \end{aligned}\right\} \quad (1)$$

where $(\tilde{r}, \tilde{\phi}, \tilde{z})$ is a system of cylindrical co-ordinates at rest, $\tilde{t}$ the time, $(\tilde{q}_r, \tilde{q}_\phi, \tilde{q}_z)$ the velocity components with respect to the co-ordinates (r, ϕ, z), $\tilde{p}$ the pressure, $\tilde{\rho}$ the density, $\tilde{\rho}_0$ the environment density, $\tilde{\omega}_0$ the angular velocity of the environment, $\tilde{r}_0$ the radius of the inner boundary of the sheet and ϵ a small parameter. Also, a suffix B refers to the basic state of equilibrium and tildes to the original physical (dimensional) quantities.

In the basic state of equilibrium, the centrifugal force is balanced by the pressure:

$$dp_B/dr = \rho r \omega^2, \quad (2)$$

where $\rho = \tilde{\rho}/\tilde{\rho}_0$ and $\omega = \tilde{\omega}/\tilde{\omega}_0$, i.e. $\rho = \omega = 1$ in the environment (regions 1 and 3 in figure 1). Integration of (2) with respect to r from origin gives

$$p_B = \left\{\begin{aligned} &p_0 + \tfrac{1}{2}r^2 \quad \text{for} \quad 0 \leqslant r \leqslant 1,\\ &p_0 + \tfrac{1}{2} + \tfrac{1}{2}\rho_1 \omega_1^2 (r^2 - 1) \quad \text{for} \quad 1 \leqslant r \leqslant 1 + \Delta r,\\ &p_0 + \tfrac{1}{2} + \tfrac{1}{2}\rho_1 \omega_1^2 \{(1+\Delta r)^2 - 1\} + \tfrac{1}{2}\{r^2 - (1+\Delta r)^2\} \quad \text{for} \quad 1 + \Delta r \leqslant r, \end{aligned}\right\} \quad (3)$$

where p_0 is the basic pressure at the origin and Δr is the thickness of the sheet.

Substitution of (1) into the basic equations of motion and neglect of terms of order ϵ^2 gives the basic equations for the disturbance:

$$0 = \partial(ur)/\partial r + \partial v/\partial \phi, \quad (4)$$

$$\frac{\partial u}{\partial t} + (\omega - 1)\frac{\partial u}{\partial \phi} - 2\omega v = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad (5)$$

$$\frac{\partial v}{\partial t} + (\omega - 1)\frac{\partial v}{\partial \phi} + 2\omega u = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi}. \quad (6)$$

In the derivation of the above equations we have used our assumption that the axial component of the wavenumber vector vanishes, i.e. that physical quantities do not depend on z.

The linearized boundary conditions on the inner and the outer interfaces of the sheet are

$$u_1 = \frac{\partial f_i}{\partial t}, \quad u_2 = \frac{\partial f_i}{\partial t} + (\omega - 1)\frac{\partial f_i}{\partial \phi}, \tag{7}$$

$$p_2 = p_1 + (1 - \rho\omega^2) f_i \tag{8}$$

on $r = 1$ and

$$u_2 = \frac{\partial f_o}{\partial t} + (\omega - 1)\frac{\partial f_o}{\partial \phi}, \quad u_3 = \frac{\partial f_o}{\partial t}, \tag{9}$$

$$p_2 = p_3 + (1 - \rho\omega^2)(1 + \Delta r) f_o \tag{10}$$

on $r = 1 + \Delta r$, where the suffixes 1, 3 and 2 refer to the inner and the outer parts of the environment and to the sheet, respectively, and f_i and f_o describe displacements of the interfaces, i.e.

$$r_i = 1 + \epsilon f_i(\phi, t), \tag{11}$$

$$r_o = 1 + \Delta r + \epsilon f_o(\phi, t). \tag{12}$$

In the above boundary conditions, (7) and (9) state that a fluid particle on the interface moves with the interface, while (8) and (10) state the continuity of the pressure across the interface. Finally, physical quantities must be finite both at the origin and at infinity (i.e. $r = \infty$).

3. Solution of the basic equations

Following the usual procedure in stability analysis, we assume that physical quantities may be expressed as

$$q = \bar{q}(r) \exp\{i(m\phi + \sigma t)\}. \tag{13}$$

Because of the single-valuedness of physical quantities, m must be an integer.

Substitution of (13) into (4)–(6) gives

$$\bar{u} = \frac{-i/\rho}{4\omega^2 - \sigma_1^2}\left\{\sigma_1 \frac{d\bar{p}}{dr} + \frac{2m\omega}{r}\bar{p}\right\}, \tag{14}$$

$$\bar{v} = \frac{1/\rho}{4\omega^2 - \sigma_1^2}\left\{\frac{m\sigma_1}{r} + 2\omega\frac{d\bar{p}}{dr}\right\}, \tag{15}$$

$$0 = \frac{d^2\bar{p}}{dr^2} + \frac{1}{r}\frac{d\bar{p}}{dr} - \frac{m^2}{r^2}\bar{p}, \tag{16}$$

where

$$\sigma_1 = \sigma + m(\omega - 1). \tag{17}$$

Solutions of (16) satisfying the boundary conditions at the origin and at infinity are

$$\bar{p}_1 = A_1 r^m, \quad \bar{p}_2 = A_2 r^m + B_2 r^{-m}, \quad \bar{p}_3 = B_3 r^{-m}. \tag{18)–(20}$$

Substitution of these solutions into boundary conditions (7)–(10) gives us a system of six linear homogeneous algebraic equations for six unknowns: A_1, A_2, B_2, B_3, f_i and f_o. Non-trivial solutions can be obtained only when the determinant of the coefficients vanishes. This condition gives us the dispersion equation

$$0 = (\sigma_1^2 - 4\omega^2)\left[\sigma^2\sigma_1^2 - \frac{(1+\Delta r)^{2m}-1}{2\rho\{(1+\Delta r)^{2m}+1\}}g\right], \tag{21}$$

or $$g = \{2\sigma - m(1-\rho\omega^2)\}\{2\sigma - m(1-\rho\omega^2) - 4\rho\omega\sigma_1\} - \sigma^4 - \rho^2\sigma_1^2(\sigma_1^2 - 4\omega^2). \tag{22}$$

Two obvious solutions of (21) are

$$\sigma_1 = \pm 2\omega. \tag{23}$$

We are not interested in these solutions, however, because they correspond to a disturbance which does not grow. Our interest is restricted to the zeros of the term in square brackets in (21). Because the sheet is thin, Δr is small in comparison with unity. In this case, the solutions can be classified into travelling and stationary disturbances with respect to our rotating system of co-ordinates. For a stationary disturbance, the root is expressed as

$$\sigma^2 = \bar{\Delta} g_0/(\omega - 1)^2, \tag{24}$$

where $$g_0 = (1-\rho\omega^2)^2 + 4\rho\omega(\omega-1)(1-\rho\omega) - m^2\rho^2(\omega-1)^4, \tag{25}$$

$$\bar{\Delta} = \{(1+\Delta r)^{2m} - 1\}/[2\rho\{(1+\Delta r)^{2m}+1\}]. \tag{26}$$

The solution of (24) is either real or purely imaginary. Therefore this disturbance does not propagate when it is unstable. This is why we call it stationary. All the disturbances of this kind become unstable if the disturbance with $m = 1$ is unstable. That is to say, the domains of instability on a ρ *vs.* ω diagram of the (stationary) disturbance with $m = 1$ constitute the union of the domains of instability for all the (stationary) disturbances. We are interested in this union and put m equal to unity in (24). This gives us

$$g_0 = -(\rho - 1)[\{2(\omega-1)^2 - 1\}\rho + 1]. \tag{27}$$

The domains of instability in figure 2 are those in which g_0 is negative. Travelling disturbances are treated by a similar procedure, and have

$$\sigma_1^2 = \frac{\bar{\Delta}\omega^2}{4(\omega-1)^2}\left(\rho - \frac{1}{\omega}\right)\left(\rho - \frac{3\omega-2}{\omega^2}\right). \tag{28}$$

The domains of instability, in which σ_1 becomes purely imaginary, are given in figure 3.

Let us next consider the angular-momentum transfer to the sheet via the disturbance excited by the above instability. Integration of the azimuthal component of the equations of motion gives us

$$\frac{\partial}{\partial \tilde{t}}\int_{\text{sheet}} \tilde{\rho}\tilde{r}^3\tilde{q}_\phi \, d\tilde{r}\, d\tilde{\phi} = -\epsilon^2\tilde{\rho}_0\tilde{r}_0^4\tilde{\omega}_0^2\int_0^{2\pi}[\rho r^2 uv]_1^{1+\Delta r}\, d\phi. \tag{29}$$

We can calculate u and v here by taking the real part of the expressions obtained for them above. We restrict ourselves to the case $m = 1$ in this procedure. Substitution of these values into (29) gives us an expression for the angular-momentum transfer. Because the calculations are lengthy though straightforward, we give the results only. For the stationary disturbance, the sign of the angular-momentum transfer is determined by the sign of the following expression:

$$I_0 = \frac{-(\rho-1)}{(\omega-1)\{\rho\omega(\omega-2)+1\}}. \tag{30}$$

The analogous expression for the travelling disturbance is

$$I_0 = -(\rho-1)(\omega-1)\{2\rho\omega^2(\omega-2)+2\omega^3-8\omega^2+12\omega-4\}. \tag{31}$$

The signs of these expressions are given in figures 2 and 3.

4. Stabilization of the instability by strong radial stratification caused by the centrifugal force

As has been discussed by many authors (see, for example, Sakurai & Matsuda 1974; Matsuda *et al.* 1975; Nakayama & Usui 1974), the effect of strong radial stratification caused by the centrifugal force is important in gas centrifuges used for the enrichment of uranium. As we stated in §1, both the travelling and the stationary disturbances are stabilized by an extremely strong stratification. Because the analysis in this case is lengthy though straightforward, we give merely an outline of the treatment, for the sake of brevity.

In the case of a compressible fluid, the density and the temperature must be treated as dependent variables in addition to those for an incompressible fluid. Instead of prescribing densities in the sheet and the environment, we prescribe temperatures in these regions, assuming that both regions are basically isothermal. The non-dimensionalization and the linearization of the basic equations are similar to those in §2. For the sake of definiteness, we assume the existence of side walls. The boundary conditions on these side walls are

$$u = 0 \quad \text{on} \quad r = r_1, r_3, \tag{32}$$

where r_1 and r_3 are the non-dimensional radii of the inner and outer side walls respectively.

The pressure distribution in the basic state of equilibrium is as follows:

$$\log(p_B/p_0) = \begin{cases} \tfrac{1}{2}\gamma M_0^2 r^2 \quad \text{for} \quad r_1 \leqslant r \leqslant 1, & (33)\\ \tfrac{1}{2}\gamma M_0^2\{1+\omega^2 T_0^{-1}(r^2-1)\} \quad \text{for} \quad 1 \leqslant r \leqslant 1+\Delta r, & (34)\\ \tfrac{1}{2}\gamma M_0^2[1+\omega^2 T_0^{-1}\{(1+\Delta r)^2-1\} \\ \qquad +r^2-(1+\Delta r)^2] \quad \text{for} \quad 1+\Delta r \leqslant r \leqslant r_3, & (35)\end{cases}$$

where $M_0^2 = \tilde{\omega}_0^2\tilde{r}_0^2/(\gamma R\tilde{T}_0)$, γ is the ratio of the specific heats, R the gas constant, $\tilde{T}_0$ the temperature of the environment and T_0 the non-dimensionalized temperature of the sheet. Boundary conditions (7) and (9) can be used as they are, while (8) and (10) are replaced by

$$p_2 = \begin{cases} p_1+\gamma M_0^2 f_i(1-\omega^2/T_0)\,p_B(r=1) \quad \text{on} \quad r=1, & (36)\\ p_3+\gamma M_0^2 f_o(1-\omega^2/T_0)(1+\Delta r)\,p_B(r=1+\Delta r) \quad \text{on} \quad r=1+\Delta r. & (37)\end{cases}$$

Substituting non-dimensional variables similar to those in (1), taking into account the above basic pressure, neglecting terms of order ϵ^2, and performing simple calculations, we obtain

$$\bar{T} = (\gamma-1)\{\bar{\rho}-i\bar{u}r\alpha_0/\sigma_1\}, \tag{38}$$

$$\bar{p} = \gamma\bar{\rho}-i\bar{u}r\alpha_0(\gamma-1)/\sigma_1, \tag{39}$$

$$\bar{v} = -\frac{mT_0\bar{\rho}}{rM_0^2\sigma_1}+\frac{i\bar{u}\omega}{\sigma_1}\{2+\alpha_1(\gamma-1)\}, \tag{40}$$

$$\bar{\rho} = \frac{i}{\sigma_1\xi}\left[r^2\frac{d\bar{u}}{dr}+r\bar{u}[1+\alpha_0r^2-\alpha_1\{2+\alpha_1(\gamma-1)\}]\right], \tag{41}$$

$$\alpha_0 = \frac{\gamma M_0^2\omega^2}{T_0}, \quad \alpha_1 = \frac{m\omega}{\sigma_1}, \quad \xi = r^2-\beta_0, \quad \beta_0 = \frac{\gamma\alpha_1^2}{\alpha_0}. \tag{42}$$

In the derivation of the above equations, we have assumed representations of physical quantities of the form (13) except for p and ρ, which are represented as

$$\left.\begin{aligned} p &= \bar{p}p_B\exp\{i(m\phi+\sigma t)\},\\ \rho &= \bar{\rho}\rho_B\exp\{i(m\phi+\sigma t)\}.\end{aligned}\right\} \tag{43}$$

Substitution of the above relations into the radial component of the equations of motion gives us

$$\begin{aligned}\phi''+\phi'^2 = {} & \{4\gamma\xi(\xi+\beta_0)\}^{-2}[\gamma^2\alpha_0^2\xi^4+2\gamma\alpha_0\{8-2\sigma_1^2/\omega^2+4(\gamma-2)\alpha_1 \\ & +(\gamma^2-2\gamma+2)\alpha_1^2\}\xi^3+\gamma^2\{-16\alpha_1+4(6-\gamma)\alpha_1^2-8(2-\gamma)\alpha_1^3 \\ & +(2-\gamma)^2\alpha_1^4\}\xi^2+4\gamma^2\beta_0\{4-4\alpha_1+(2-\gamma)\alpha_1^2\}\xi+12\gamma^2\beta_0^2],\end{aligned} \tag{44}$$

where we have used the following transformations motivated by the WKB method of approximation:

$$\bar{u} = iu_2e^{\phi}, \quad u_2 = \xi^{\frac{1}{2}}(\xi+\beta_0)^{-1}\exp(-\tfrac{1}{4}\alpha_0\xi). \qquad (45),(46)$$

Corresponding to the case of extremely strong stratification caused by the centrifugal force, we restrict ourselves to a large value of M_0 and assume

$$\sigma = M_0^{-1}\bar{\sigma}, \quad \bar{\sigma} \sim 1 \tag{47}$$

to analyse stationary disturbances. Introduction of the above assumptions and neglect of terms of order M_0^{-1} gives us approximate versions of (44):

$$\phi'^2 = (\tfrac{1}{4}\alpha_0)^2\{\xi+(\gamma-2)^2\beta_0\gamma^{-2}\}/(\xi+\beta_0) \tag{48}$$

for regions 1 and 3 and

$$\phi'^2 = (\tfrac{1}{4}\alpha_0)^2\xi^2/(\xi+\beta_0)^2 \tag{49}$$

for region 2, where we have used the fact that

$$\alpha_0 \sim M_0^2, \quad \alpha_1 \sim M_0, \quad \beta_0 \sim 1$$

in regions 1 and 3 and the fact that

$$\alpha_0 \sim M_0^2, \quad \alpha_1 \sim 1, \quad \beta_0 \sim M_0^{-2}$$

in region 2.

The solutions of (48) are

$$\bar{u}_i = \frac{\xi_1^{\frac{1}{2}}}{\xi_1+\beta_{01}} \exp\left(-\tfrac{1}{4}\alpha_{01}\xi_1\right)\{a_{i1}\exp\phi_1 + a_{i2}\exp\left(-\phi_1\right)\} \quad (i = 1, 3), \tag{50}$$

$$\phi_1 = \tfrac{1}{4}\alpha_{01}[(\xi_1+\beta_{01})^{\frac{1}{2}}(\xi_1+\bar{a}\beta_{01})^{\frac{1}{2}} - \beta_{01}(1-\bar{a})\log\{(\xi_1+\bar{a}\beta_{01})^{\frac{1}{2}} + (\xi_1+\beta_{01})^{\frac{1}{2}}\}], \tag{51}$$

where

$$\alpha_{01} = \gamma M_0^2, \quad \beta_{01} = \frac{m^2}{M_0^2\sigma^2}, \quad \bar{a} = \left(\frac{\gamma-2}{\gamma}\right)^2, \quad \xi_1 = r^2-\beta_{01} \tag{52}$$

and the suffix i refers to regions 1 and 3. The solution of (49) is

$$\bar{u}_2 = \frac{\xi^{\frac{1}{2}}}{\xi+\beta_0}\exp\left(-\tfrac{1}{4}\alpha_0\xi\right)\{a_{21}\exp\phi_2 + a_{22}\exp\left(-\phi_2\right)\}, \tag{53}$$

$$\phi_2 = \tfrac{1}{4}\alpha_0\{\xi - \beta_0\log\left(\xi+\beta_0\right)\}, \tag{54}$$

where

$$\alpha_0 = \frac{\gamma M_0^2\omega^2}{T_0}, \quad \beta_0 = \frac{T_0}{M_0^2(\omega-1)^2}, \quad \alpha_1 = \frac{\omega}{\omega-1}, \quad \xi = r^2-\beta_0. \tag{55}$$

Solutions (50) and (53) each contain three unknown constants. Substitution of these solutions into the eight boundary conditions (7), (9), (32), (36) and (37) provides us with a system of eight linear homogeneous algebraic equations for the above six constants plus f_i and f_o. Non-trivial solutions can be obtained only when the determinant of the coefficients vanishes. This gives us a dispersion equation from which σ may be determined. The calculations are lengthy though straightforward, and give us the following approximate representation of the dispersion equation:

$$(1-\bar{a})\,m^2/M_0^2\sigma^2 = 1-(\omega^2G/\gamma T_0)^2\Delta r^2, \tag{56}$$

where G is a factor of order unity. In the derivation of (56), we have used the smallness of Δr and the fact that the radial scale height is small in comparison with the radial distances from the sheet to the side walls. Because $\bar{a} < 1$, σ is real for every value of ω and T_0 of order unity.

Finally, to treat travelling disturbances, we assume

$$\sigma_1 = \bar{\sigma}_1 M_0^{-1}, \quad \bar{\sigma}_1 \sim 1. \tag{57}$$

The rest of the procedure is completely analogous to that for the stationary disturbances. The solution σ_1 of the dispersion equation is again real for every value of ω and T_0 of order unity. This corresponds to stabilization of both the travelling and the stationary disturbances by the strong radial stratification caused by the centrifugal force.

5. Concluding remarks

In conclusion, let us examine the validity of our approximations. A standard gas centrifuge has radius 10 cm and height 100 cm and rotates at 40000 r.p.m. The circular slit in the lid has inner radius 6 cm and width 1 cm. The volume flux of operating gas through this slit is 20 m³/min. The operating gas is uranium hexafluoride (UF_6), whose gas constant is $2{\cdot}36\times10^5$ erg °C^{-1} mole^{-1}. The temperature of the gas is 50 °C and the sound velocity in it 90 m/s. The pressure at the boundary is kept at 100 mmHg at most in order to be safe from liquefaction of the UF_6. The kinematic viscosity of UF_6 corresponding to these boundary conditions is 0·115. Finally, the ratio of specific heats is 1·06.

On the basis of the above data, our parameters M^2 and Δr are 8·3 and 0·17, respectively. Our restriction to cases with large M^2 and small Δr is thus reasonable. Discontinuities in angular velocity and temperature between the sheet and the environment are smoothed out by Stewartson $E^{\frac{1}{3}}$- and $E^{\frac{1}{4}}$-layers. Of these, $E^{\frac{1}{4}}$-layer on the inner interface is the thickest, its thickness being about 1 cm. Because this is of the same order of magnitude as the width of the sheet, our inviscid approximation in which interfaces are treated as discontinuities must be amended. Finally, the axial velocity corresponding to the above volume flux rate is 81·6 m/s. A fluid particle with this axial velocity travels from the top to the bottom in about 0·01 s. Because this is equal to eight rotation periods, which is small in comparison with the time scales discussed in § 4, it is necessary to take into account the effect of the axial flow.

Attempts to improve our present treatment according to the above discussion are outside the scope of this paper.

REFERENCES

BUSSE, F. H. 1968 Shear flow instabilities in rotating systems. *J. Fluid Mech.* **33**, 577–589.

GANS, R. F. 1975 On the stability of shear flow in a rotating gas. *J. Fluid Mech.* **68**, 403–412.

HASHIMOTO, K. 1975 A source–sink flow of an incompressible rotating fluid. *J. Phys. Soc. Japan*, **38**, 1508–1515.

HASHIMOTO, K. 1976 On the stability of the Stewartson layer. *J. Fluid Mech.* **76** (in press).

HIDE, R. & TITMAN, C. W. 1967 Detached shear layers in a rotating fluid. *J. Fluid Mech.* **29**, 39–60.

HOWARD, L. N. & GUPTA, A. S. 1962 On the hydrodynamic and hydromagnetic stability of swirling flows. *J. Fluid Mech.* **14**, 463–476.

MASLOWE, S. A. 1974 Instability of rigidly rotating flows to non-axisymmetric disturbances. *J. Fluid Mech.* **64**, 307–317.

MATSUDA, T., SAKURAI, T. & TAKEDA, H. 1975 Source–sink flow in a gas centrifuge. *J. Fluid Mech.* **69**, 197–208.

NAKAYAMA, W & USUI, S. 1974 Flow in rotating cylinder of a gas centrifuge. *J. Nucl. Sci. Tech.* **11**, 242–262.

PEDLEY, T. J. 1968 On the stability of rapidly rotating shear flows to non-axisymmetric disturbances. *J. Fluid Mech.* **31**, 603–607.

PEDLEY, T. J. 1969 Oh the instability of viscous flow in a rapidly rotating pipe. *J. Fluid Mech.* **35**, 97–115.

SAKURAI, T. 1975 Baroclinic-type instability in a gas centrifuge heated from above. *J. Fluid Mech.* **72**, 321–329.

SAKURAI, T. & MATSUDA, T. 1974 Gasdynamics of a centrifugal machine. *J. Fluid Mech.* **62**, 727–736.

SIEGMANN, W. L. 1974 Evolution of unstable shear layers in a rotating fluid. *J. Fluid Mech.* **64**, 289–305.

STARR, V. P. 1968 *Physics of Negative Viscosity Phenomena*, chaps. 2 and 3. McGraw-Hill.